

# Systematics of One-Loop Scattering Amplitudes in $N = 4$ Super Yang-Mills Theories

---

**Mingxing Luo and Congkao Wen**

*Zhejiang Institute of Modern Physics, Department of Physics  
Zhejiang University, Hangzhou, Zhejiang 310027, P R China  
E-mail: luo@zimp.zju.edu.cn, wenagua@sina.com*

**ABSTRACT:** One-loop scattering amplitudes in  $N = 4$  super Yang-Mills (SYM) theories are analyzed in the paradigm of maximal helicity violating Feynman diagrams. There are very limited number of loop integrals to be evaluated. For a process with  $n$  external particles, there are only  $[n/2] - 1$  generically independent integrals. Furthermore, the relations between leading  $N_c$  amplitudes  $A_{n;1}$  and sub-leading amplitudes  $A_{n;c}$  are found to be identical to those obtained from conventional field theory calculations, which can be interpreted as an indirect support for the paradigm.

**KEYWORDS:** MHV, One-loop, Supersymmetry, Yang-Mills Theory.

---

## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. Review of existing results and the paradigm of MHV Feynman diagrams</b>	<b>2</b>
<b>3. Classification of MHV Feynman diagrams</b>	<b>3</b>
<b>4. The relation between <math>A_{n;1}</math> and <math>A_{n;c}</math></b>	<b>6</b>
4.1 The case of 1PI MHV Feynman diagrams	6
4.2 The case of 1PR MHV Feynman diagrams	8
<b>5. Conclusion</b>	<b>9</b>

---

## 1. Introduction

In perturbative Yang-Mills theories, the Parke-Taylor formula for maximal helicity violating (MHV) amplitudes summarizes much of its elegance [1]. For theories including supersymmetries, some of the salient features persist beyond the tree level [2, 3]. Over the years, various techniques were developed to elucidate these properties. Among them, the color decomposition method and the spinor helicity technique [4] have been proved to be extremely useful and efficient [5].

Recently, Witten pointed out a deep relation [6] between  $N = 4$  SYM theories and one type B topological string theory, by re-expressing SYM scattering amplitudes in the language of twistor theories [7]. Taking advantage of insights thus gained and by a careful analysis of known helicity amplitudes, Cachazo, Svrcek, and Witten (CSW) [8] proposed a novel prescription to calculate tree level amplitudes, which uses the MHV amplitudes as vertices to construct all other amplitudes, thus initiated the paradigm of MHV Feynman diagrams. The efficiency of the method is phenomenal and the validity of the method has been checked by various tree level calculations [9, 10, 11, 12, 13, 14, 15, 16, 17]. In [18], the method was extended to the calculation of one-loop MHV amplitudes and results from conventional quantum field theory were reproduced in the large  $N_c$  limit. It turns out that the paradigm is independent of the large  $N_c$  approximation, at least in the case of MHV scattering amplitudes to one-loop [19]. The twistor-space structure of one-loop amplitudes are further studied in [20, 21, 22, 23]. On the other hand, tree-level amplitudes were also obtained from connected curves in twistor string theories [24, 25, 26].

In this paper, one-loop MHV Feynman diagrams will be classified. It turns out there are only very limited number of integrals to be evaluated, though there are numerous loop diagrams. For a process with  $n$  external particles, there are only  $[n/2] - 1$  generically

independent integrals. These integrals depend crucially on the number of external particles of negative helicity, but marginally on  $n$ . All loop integrals can be obtained from these generic integrals by appropriately substituting relevant helicities and momenta. We then prove that the relation between leading  $N_c$  amplitudes  $A_{n;1}$  and sub-leading amplitudes  $A_{n;c}$  are identical to those obtained from conventional field theory calculations [2].

As there are scant one-loop calculations beyond  $q = 2$  in the paradigm, these results are speculative to certain extent. But given their regularity, these results might merit presentation. Given the strong evidences gained so far, it is hard to imagine the paradigm would go astray, at least at the one-loop level. Actually, the relation between  $A_{n;1}$  and  $A_{n;c}$  as revealed could be interpreted as an indirect support for the paradigm.

The paper will be organized as the follows. In section 2, we review existing one-loop results and give a brief introduction to the paradigm of MHV Feynman diagrams. In section 3, we give a classification of one-loop MHV Feynman diagrams. In section 4, we establish the relation between leading  $N_c$  amplitudes  $A_{n;1}$  and sub-leading amplitudes  $A_{n;c}$ . We conclude in section 5.

## 2. Review of existing results and the paradigm of MHV Feynman diagrams

In four dimensional space-time, a momentum  $k_\mu$  can be expressed as a bispinor  $k_{a\dot{a}} = k_\mu \sigma_{a\dot{a}}^\mu$ . For massless particle,  $k^2 = 0$ , the momentum can be factorized  $k_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$  in terms of spinors  $\lambda_a, \tilde{\lambda}_{\dot{a}}$  of positive and negative chirality. Spinor products are defined to be  $\langle \lambda_1, \lambda_2 \rangle = \epsilon_{ab} \lambda_1^a \lambda_2^b$  and  $\langle \tilde{\lambda}_1, \tilde{\lambda}_2 \rangle = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$ , which are usually abbreviated as  $\langle 1, 2 \rangle$  and  $[1, 2]$ .

At tree level, the scattering amplitudes of  $n$  gluons with one or none opposite type of helicity vanish. The amplitudes with two negative helicity are called maximally helicity violating (MHV) amplitudes. For  $N = 4$  SYM theories, a MHV amplitude is given by the generalized Parke-Taylor formula which includes particles of all helicity [27]:

$$i(2\pi)^4 \delta^{(4)} \left( \sum_{i=1}^n \lambda_i \tilde{\lambda}_i \right) \delta^{(8)} \left( \sum_{i=1}^n \lambda_i \eta^i \right) A_n(\{k_i, \lambda_i, a_i\}) \quad (2.1)$$

where  $\eta_A^i$  are anti-commuting variables,  $A$  is an index of the anti-fundamental representation of  $SU(4)$ ;  $k_i, \lambda_i$ , and  $a_i$  are the momentum, helicity, and the color index of the  $i$ -th external particles, respectively;

$$A_n(\{k_i, \lambda_i, a_i\}) = \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) \prod_{i=1}^n \frac{1}{\langle \sigma(i), \sigma(i+1) \rangle} \quad (2.2)$$

where  $S_n/Z_n$  is the set of non-cyclic permutation of  $\{1, \dots, n\}$ . The  $U(N_c)$  generators  $T^a$  are the set of hermitian  $N_c \times N_c$  matrices normalized such that  $\text{Tr}(T^a T^b) = \delta^{ab}$ . Here and after the gauge coupling constant  $g$  is not included, but can be easily recovered when needed. The supersymmetric amplitudes can be expanded in powers of the  $\eta_A^i$ , and each term of this expression corresponds to a particular scattering amplitude.

To one-loop, the results take the form [2]

$$A_n^{1\text{-loop}}(\{k_i, \lambda_i, a_i\}) = \sum_{c=1}^{[n/2]+1} \sum_{\sigma \in S_n/S_{n;c}} Gr_{n;c}(\sigma) A_{n;c}(\sigma) \quad (2.3)$$

where  $S_n$  is the set of all permutation of  $n$  objects and  $S_{n;c}$  is the subset leaving  $Gr_{n;c}$  invariant. The color factors are

$$\begin{aligned} Gr_{n;1} &= N_c \text{Tr}(T^{a_1} \dots T^{a_n}) \\ Gr_{n;c} &= \text{Tr}(T^{a_1} \dots T^{a_{c-1}}) \text{Tr}(T^{a_c} \dots T^{a_n}) \end{aligned} \quad (2.4)$$

$A_{n;c>1}$  can be obtained by performing appropriate sums over permutations of  $A_{n;1}$ :

$$A_{n;c}(1, \dots, c-1; c, \dots, n) = (-)^{c-1} \sum_{\sigma \in \text{COP}\{\alpha\}\{\beta\}} A_{n;1}(\sigma) \quad (2.5)$$

where  $\alpha_i \in \{\alpha\} \equiv \{c-1, \dots, 1\}$ ,  $\beta_i \in \{\beta\} \equiv \{c, c+1, \dots, n\}$  and  $\text{COP}\{\alpha\}\{\beta\}$  is the set of all permutations of  $\{1, \dots, n\}$  with  $n$  held fixed that preserve the cyclic ordering of the  $\alpha_i$  within  $\{\alpha\}$  and of the  $\beta_i$  within  $\{\beta\}$ , while allowing for all possible relative ordering of the  $\alpha_i$  with respect to the  $\beta_i$ .

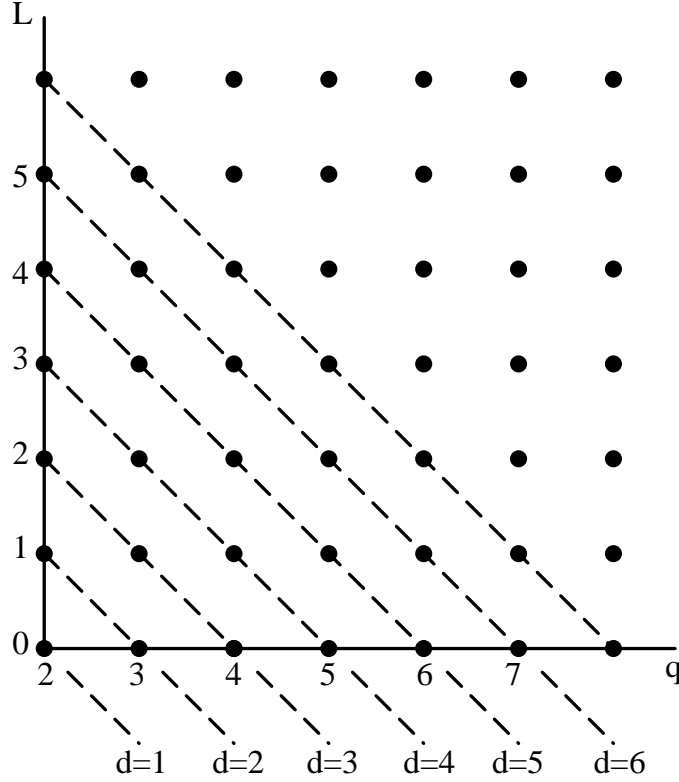
In [8], it was proposed that these MHV amplitudes can be continued to off-shell and be used as vertices to generate tree diagrams for all non-MHV amplitudes. These new diagrams are dubbed as MHV diagrams. Including all relevant factors, the  $n$ -points MHV vertex is

$$\mathcal{V}_n = i \text{Tr}(T^{a_1} \dots T^{a_n}) (2\pi)^4 \delta^{(4)} \left( \sum_{i=1}^n \lambda_i \tilde{\lambda}_i \right) \delta^{(8)} \left( \sum_{i=1}^n \lambda_i \eta^i \right) \prod_{i=1}^n \frac{1}{\langle i, i+1 \rangle} \quad (2.6)$$

The propagator for internal lines of momentum  $k$  is simply taken as  $i/k^2$ . The issue is how to define the corresponding spinor for particles with off-shell momentum  $k$ . The CSW prescription is to take an arbitrary spinor of positive helicity  $\tilde{\xi}$  and define  $\lambda_a = k_{a\dot{a}} \tilde{\xi}^{\dot{a}}$ . These rules are extended for applications in one-loop [18]. To the leading order of  $N_c$ ,  $A_{n;1}$  is consistent with the result of conventional quantum field was reproduced in [18]. One-loop sub-leading results were reproduced in [19].

### 3. Classification of MHV Feynman diagrams

A MHV vertex can be represented by a dot with certain number of lines attached. Each line represents an external or internal particle and there can only be two lines with negative helicity on each vertex. It is convenient to generate quiver diagrams by linking certain number of dots together. The links represent internal particles. MHV Feynman diagrams are generated by distributing external lines on the dots according to color and helicity structures. Parallel to conventional quantum field theory, we will call a MHV diagram as one-particle-irreducible (1PI) if the diagram cannot be broken into two pieces by cutting one of its internal lines, as one-particle reducible (1PR) if otherwise. Note there is a difference



**Figure 1:** Graphic representation of Eq. (3.1), where  $d$  is the degree of the algebraic curve,  $q$  the number of external particles of negative helicity, and  $l$  the number of loops. Only the lowest line  $l = 0$  and the point  $(q, l) = (2, 1)$  are relatively well understood.

between 1PI MHV Feynman diagrams and 1PI conventional Feynman diagrams. 1PI MHV Feynman diagrams usually do not correspond to 1PI conventional Feynman diagrams.

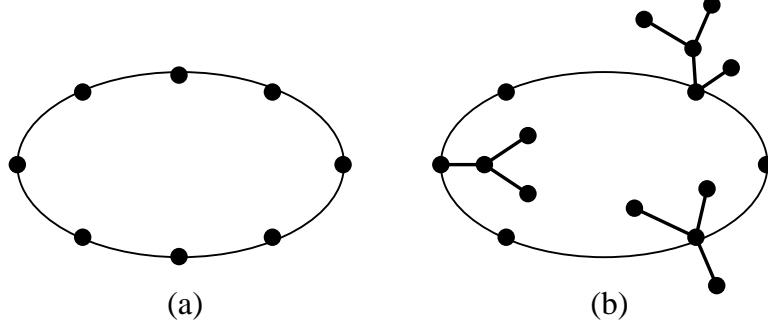
It was conjectured in [6] that scattering amplitudes in  $N = 4$  SYM theories are supported on certain algebraic curves, if expressed in the twistor-space. For a process with  $q$  particles of negative helicity, the degree of the corresponding algebraic curve is

$$d = q - 1 + l \quad (3.1)$$

where  $l$  is the number of loops.

Note that  $d$  is independent of the total number of external particles in consideration. In the paradigm of MHV Feynman diagrams,  $d$  is the number of MHV vertices in a given diagram. Graphically expressed (Figure 1), Eq (3.1) serves as a good organization principle in the paradigm. The horizontal line  $l = 0$  represents all tree diagrams, one point for a specific  $q$ . The vertical line  $q = 2$  represents all loop diagrams for the MHV scattering amplitudes, one point for a specific  $l$ . For each point on line  $l = 0$ , the corresponding quiver diagrams are generated by making different connected trees of  $d$  points in all possible manner. From which one can write out the amplitude in a straightforward manner, though

it takes efforts to convert them to familiar forms. Beyond this line, only the point  $(q, l) = (2, 1)$  has been worked out analytically.



**Figure 2:** (a) One loop quiver diagrams: (a) 1PI; (b) 1PR.

We will call MHV diagrams of  $d$  vertices as degree  $d$  MHV diagrams. At one-loop, things simplifies significantly.  $d$  is then equal to  $q$ , the number of external particles with negative helicity. And degree  $d$  1PI MHV quiver diagram is a circle with  $d$ -dots on it (Figure 2a). The lowest degree  $d$  is two, corresponding to the MHV amplitudes, which is also irreducible. In general, degree  $d$  one-loop MHV diagrams is reducible. It includes one degree  $d$  1PI MHV diagrams and 1PR MHV diagrams made of degree  $k = \{2, \dots, d-1\}$  1PI MHV diagrams by attaching trees of total degree  $d-k$  in all topologically inequivalent ways (Figure 2b).

MHV Feynman diagrams can also be classified according to their color structures, which is closely related to their topologies. One-loop MHV diagrams with all external lines outside of the circle will be referred to as leading  $N_c$  MHV diagrams, since they give the  $G_{n;1}$  color factors; those with external lines both outside and inside the circle as sub-leading MHV diagrams, since they give the  $G_{n;c>1}$  color factors. (One-loop MHV diagrams with all external lines inside of the circle are identical to leading MHV diagrams, up to possible overall signs.) In the next section, it will be proved that all sub-leading diagrams can be expressed as sums of the leading ones. Thus if the loops are calculated iteratively in  $d$ , one encounters one generically new quiver loop diagram for each added degree. For a process with  $n$  external particles, there are  $n-3$  generically different quiver loop diagrams in total.

For a given color factor, one gets all MHV diagrams from relevant quiver diagrams, by distributing external particles on the vertices in all possible manners but keeping the relative order according to the color indices. Specifically, for a process with  $n$  external particles and with a degree of  $d$ , we first put  $m_k$  external lines on the  $k$ -th vertex,  $\sum_{k=1}^d m_k = n$ . For each set  $\{m_k\}$ , we put on the external lines clock-wise according to the color indices to get a diagram, and use cyclic permutation to get all different ones:

$$A_n = \sum_{\{m_k\}} \sum_{\sigma \in (\{\alpha\})} \int \prod_{i=1}^d \frac{d^4 L_i}{L_i^2} d^4 \eta_i \prod_{k=1}^d \mathcal{V}_{m_k+2}(\sigma) \quad (3.2)$$

where  $\mathcal{V}_{m_k+2}$  is the  $m_k + 2$ -point vertex function given in Eq (2.6),  $L_i$  is the  $i$ -th internal line,  $\eta_i$  the related fermionic variable, and  $(\{\alpha\})$  is the set of all cyclic permutation of  $n$  objects. Obviously, for a given degree  $d$ , the 1PR integrals can be obtained from those 1PI integrals of degrees less than  $d$ . For 1PI cases of degree  $d$ , there are numerous integrals which are generated by distributing external lines on the quiver diagram in all possible ways. It turns out that all relevant loop integrals can be obtained from one generic integral by appropriately substituting relevant helicities and momenta. These integrals depend on  $d$  crucially but only marginally on  $n$ . More so, for  $q > [(n+1)/2]$ , the result can be obtained from the case of  $q' = n - q$  via parity inversion. Thus only  $[n/2] - 1$  of the integrals need to be evaluated eventually. This trend seems to persist to higher loops, reflecting an intrinsic property implicated by Eq (3.1). We note in passing that the leading  $N_c$  term of MHV scattering amplitudes at all orders seem to retain their tree level helicity structures, provided that the paradigm of MHV Feynman diagrams is also valid to all orders. There are also regularities in sub-leading orders which need to be further studied [28].

#### 4. The relation between $A_{n;1}$ and $A_{n;c}$

In this section we will prove that  $A_{n;1}$  and  $A_{n;c}$  in the paradigm of MHV Feynman diagrams are related in the same way as those in conventional quantum field theory. The first part is for 1PI MHV diagrams, which is a generalization of the proof in section 4 of [19] and includes MHV scattering amplitudes as special cases of  $d = 2$ . The second part deals with 1PR diagrams.

##### 4.1 The case of 1PI MHV Feynman diagrams

One-loop 1PI MHV Feynman diagrams are generated from quiver diagrams as the one in Figure 2a. We first give the prescription to obtain the leading contribution. For a given color factor,  $\text{Tr}(T^{a_1} \dots T^{a_n})$ , one gets

$$A_{n;1}^{\text{1PI}} = \sum_{\{m_k\}} \sum_{\sigma \in (\{\alpha\})} \int \prod_{k=1}^d \frac{d^4 L_k}{L_k^2} d^4 \eta_k \delta^{(4)}(L_k - L_{k-1} + P_k) \delta^{(8)}(\Theta_k) F(\{m_k\}, \sigma) \quad (4.1)$$

where  $P_k$  is the total momentum of all external particles on the  $k$ -th MHV vertex,  $L_k$  is the internal momentum between the  $k$ -th and  $(k+1)$ -th vertices,  $\eta_k$  is the related fermionic variables,  $\Theta_k = l_k \eta_k - l_{k-1} \eta_{k-1} + \sum_{i_k=c_k+1}^{c_{k+1}} \lambda_{i_k} \eta_{i_k}$ ,  $c_k = \sum_{p=1}^{k-1} m_p$ , and

$$F(\{m_k\}, \sigma) = \prod_{k=1}^d \frac{1}{\langle l_k, l_{k-1} \rangle \langle l_{k-1}, \sigma(c_k+1) \rangle \langle \sigma(c_{k+1}), l_k \rangle} \prod_{i_k=c_k+1}^{c_{k+1}-1} \frac{1}{\langle \sigma(i_k), \sigma(i_k+1) \rangle}. \quad (4.2)$$

For a specific sub-leading term with the color factor <sup>1</sup>

$$\text{Tr}(T^{a_{c-1}} \dots T^{a_1}) \text{Tr}(T^{a_c} \dots T^{a_n}) \quad (4.3)$$

---

<sup>1</sup>Differing from [19], we here reverse the order of indices in the first trace.

we construct the sub-leading term  $A_{n;c}(c-1, \dots, 1; c, \dots, n)$  according to Eq (2.5). A lengthy but straightforward calculation gives

$$A_{n;c}^{\text{1PI}} = (-1)^{c-1} \sum_{\{m_k\}} \sum_{\sigma \in \text{COP}''\{\alpha\}\{\beta\}} \int \prod_{k=1}^d \frac{d^4 L_k}{L_k^2} d^4 \eta_k \times \delta^{(4)}(L_k - L_{k-1} + P_k) \delta^{(8)}(\Theta_k) F(\{m_k\}, \sigma) \quad (4.4)$$

where  $\alpha_i \in \{\alpha\} \equiv \{1, \dots, c-1\}$ ,  $\beta_i \in \{\beta\} \equiv \{c, c+1, \dots, n\}$  and  $\text{COP}''\{\alpha\}\{\beta\}$  is the set of all permutations of  $\{1, \dots, n\}$  that preserve the cyclic ordering of  $\alpha_i$  within  $\{\alpha\}$  and of  $\beta_i$  within  $\{\beta\}$ , while allowing for all possible relative ordering of  $\alpha_i$  with respect to  $\beta_i$ .

In the rest of the section, we will show that this  $A_{n;c}(c-1, \dots, 1; c, \dots, n)$  can also be obtained from all relevant sub-leading MHV diagrams. Explicitly, these sub-leading MHV diagrams are generated by distributing the first  $c-1$  external particles inside the circle, and the rest  $n-c+1$  external particles outside the circle, but keeping the cyclic order inside and outside of the circle, respectively.

We now take a specific diagram with  $m_k$  external lines on the  $k$ -th vertex, among them,  $j_k$  of which inside the circle and  $i_k$  of which outside of the circle,  $m_k = i_k + j_k$ , as shown on the left of Figure 3. Denote  $I_k = \sum_{p=1}^{k-1} i_p$ ,  $J_k = \sum_{p=1}^{k-1} j_p$ . Applying Eq (4.6) in [19], we can make the following transformation for graphs on the left side of Figure 3 to its right side:

$$\begin{aligned} & \prod_{k=1}^d \left[ \left( \prod_{r=c+I_{k+1}}^{c+I_{k+1}-1} \frac{1}{\langle \beta_r, \beta_{r+1} \rangle} \right) \frac{1}{\langle \beta_{c+I_{k+1}}, l_k \rangle \langle l_k, \alpha_{J_{k+1}} \rangle} \right. \\ & \times \left. \left( \prod_{r=J_{k+1}-1}^{J_k+1} \frac{1}{\langle \alpha_{r+1}, \alpha_r \rangle} \right) \frac{1}{\langle \alpha_{J_k+1}, l_{k-1} \rangle \langle l_{k-1}, \beta_{c+I_k+1} \rangle} \right] \\ & = (-1)^{c-1} \prod_{k=1}^d \left[ \frac{1}{\langle l_k, l_{k-1} \rangle} \sum_{\sigma \in \text{COP}'\{\alpha^{(k)}\}\{\beta^{(k)}\}} \frac{1}{\langle l_{k-1}, \sigma(c_k+1) \rangle} \right. \\ & \times \left. \left( \prod_{i_k=c_{k+1}}^{c_{k+1}-1} \frac{1}{\langle \sigma(i_k), \sigma(i_k+1) \rangle} \right) \frac{1}{\langle \sigma(c_{k+1}), l_k \rangle} \right] \quad (4.5) \end{aligned}$$

where  $\alpha_{i_k} \in \{\alpha^{(k)}\} \equiv \{J_k+1, \dots, J_{k+1}\}$ ,  $\beta_{i_k} \in \{\beta^{(k)}\} \equiv \{c+I_k, \dots, c+I_{k+1}\}$ ,  $\text{COP}'\{\alpha^{(k)}\}\{\beta^{(k)}\}$  is the set of all permutations of  $\{1, \dots, m_k\}$  that preserve the ordering of  $\alpha_{i_k}$  within  $\{\alpha^{(k)}\}$  and of  $\beta_{i_k}$  within  $\{\beta^{(k)}\}$ , while allowing for all possible relative ordering of  $\alpha_{i_k}$  with respect to  $\beta_{i_k}$ . Cyclically permuting all elements in  $\{\alpha\} = \prod_{k=1}^d \bigcup \{\alpha^{(k)}\}$  and  $\{\beta\} = \prod_{k=1}^d \bigcup \{\beta^{(k)}\}$ , take the sum for all cases of  $i_k$ ,  $j_k$  and  $m_k$ , and including all the  $\delta$  functions, we get

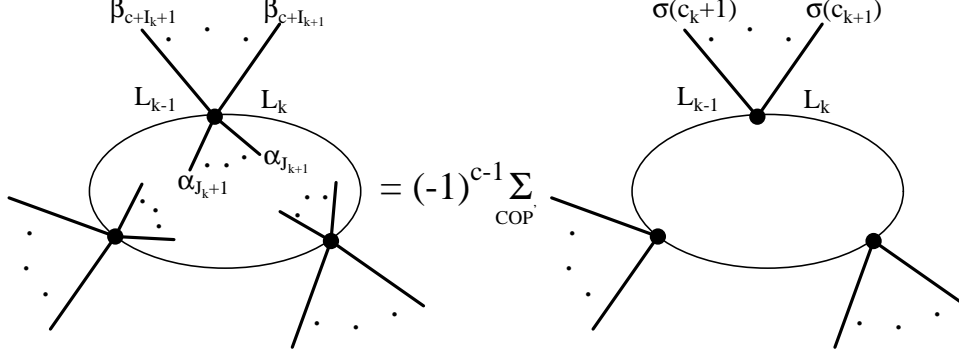
$$\sum_m (-1)^{c-1} \sum_{\sigma \in \text{COP}''\{\alpha\}\{\beta\}} \delta^{(4)}(L_k - L_{k-1} + P_k) \delta^{(8)}(\Theta_k) F(\{m_k\}, \sigma) \quad (4.6)$$

which is nothing but the integrand in Eq (4.4).

Thus for 1PI MHV Feynman diagrams, we have proved that sub-leading amplitudes  $A_{n;c}$  obtained from the sub-leading MHV diagrams are related to the leading  $N_c$  amplitudes



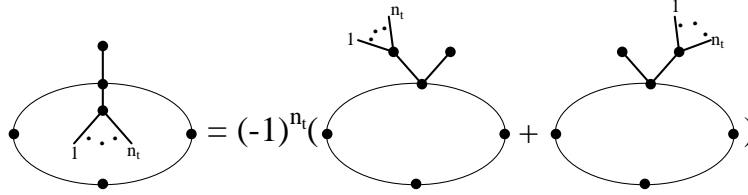
$A_{n;1}$  in the same way as those obtained from conventional field theory calculations. In particular, this relation does not assume any of the external particles to be on-shell, since the Schouten identity and our reasoning have not put any requirement on the spinors.



**Figure 3:** Graphic representation of Eq (4.5), a one-loop sub-leading MHV diagram is expressed in terms of a set of one-loop leading MHV diagrams. All the external lines inside the circle are reflected to outside, in the manner prescribed in Eq (4.6) and Figure 7 of [19].

#### 4.2 The case of 1PR MHV Feynman diagrams

One encounters two subtleties in cases of 1PR MHV diagrams. The first subtlety is that there could be trees both inside and outside the circle. An inside tree can be reflected to the outside, in almost the same manner as a single external line, with two modifications: the order of external lines on the inside the tree has to be kept as if it is a mirror reflection, and the overall sign is  $(-1)^{n_t}$  rather than  $-1$ , where  $n_t$  is the total number of external lines on the tree. Each tree (inside and outside) is counterated as a single line in the permutation of  $COP'\{\alpha\}\{\beta\}$ . Shown in Figure 4 is the special case of one tree inside and one tree outside the circle. Complicated cases can be constructed in similar fashion.

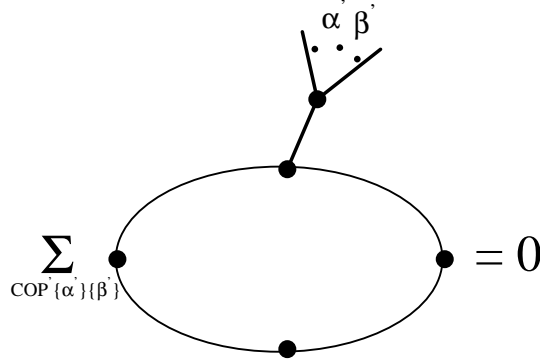


**Figure 4:** Reflecting a tree inside the circle to the outside. Note the order of the external lines after the reflection and the overall sign.

The second subtlety is that there are terms in  $A_{n;c}$  constructed from  $A_{n;1}$  which are proportional to

$$\sum_{\sigma \in COP'\{\alpha'\}\{\beta'\}} \frac{1}{\langle l, \sigma(1) \rangle} \left( \prod_{i=1}^{n-1} \frac{1}{\langle \sigma(i), \sigma(i+1) \rangle} \right) \frac{1}{\langle \sigma(n), l \rangle} \quad (4.7)$$

where  $\alpha'/\beta'$  indicate a subset of internal lines originally inside/outside of the circle, with at least one set of them on a tree attached to the circle. Graphically, it is shown in Figure 5. Obviously one cannot obtain these term from the sub-leading MHV diagrams. Fortunately these terms vanish upon summation over  $\sigma$ , thanks to the Schouten's identity again.



**Figure 5:** Graphic representation of Eq (4.7), indicating terms from  $A_{n;c}$  constructed out of  $A_{n;1}$  with the lines originally inside the circle mingle with those originally outside.

With these two subtleties taken case of, the rest of the proof follows exactly the same steps as those in the case of 1PI MHV diagrams.

## 5. Conclusion

In this paper, we have analyzed general amplitudes in general  $N = 4$  SYM theories to full one-loop in the paradigm of MHV Feynman diagrams. We classified all one-loop MHV Feynman diagrams. It turns out there are only very limited number of integrals to be evaluated, though there are numerous loop diagrams. For a process with  $n$  external particles, there are only  $[n/2] - 1$  generically independent integrals. For a specific case with  $q < [(n + 1)/2]$  negative helicities, only  $q - 1$  generically independent integrals need to be evaluated. More so, the first  $q - 2$  integrals can be obtained from calculations for processes with lesser number of negative helicities. For a specific case with  $q > [(n + 1)/2]$  negative helicities, the result can be obtained from the case of  $q' = n - q$  via parity inversion. These integrals depend crucially on  $q$  but marginally on  $n$ . All loop integrals can be obtained from these generic integrals by appropriately substituting relevant helicities and momenta. We have also prove that the relation between leading  $N_c$  amplitudes  $A_{n;1}$  and sub-leading amplitudes  $A_{n;c}$  are identical to those obtained from conventional field theory calculations [2].

As there are scant one-loop calculations beyond  $q = 2$  in the paradigm, these results are speculative to certain extent. But given the strong evidence obtained so far, it is difficult to imagine the paradigm would go astray, though an understanding of the approach from the perspective of conventional quantum field theory is still wanting. Actually, the relation between  $A_{n;1}$  and  $A_{n;c}$  as revealed in this paper could be interpreted as an indirect evidence

to support the paradigm. We suspect that the integrals will come out to be right. It is now imperative to devise an efficient way to calculate them.

## Acknowledgments

This work is supported in part by the National Science Foundation of China.

## References

- [1] S. Parke and T. Taylor, “An Amplitude For N Gluon Scattering”, Phys. Rev. Lett. 56, 2459 (1986); F. A. Berends and W. T. Giele, “Recursive Calculations For Processes With N Gluons”, Nucl. Phys. B306, 759 (1988).
- [2] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, “One Loop N Point Gauge Theory Amplitudes, Unitarity And Collinear Limits”, Nucl. Phys. B 425, 217 (1994), hep-ph/9403226.
- [3] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, “Fusing Gauge Theory Tree Amplitudes into Loop Amplitudes”, Nucl. Phys. B 435, 59 (1995), hep-ph/9409265.
- [4] Z. Xu, D. H. Zhang and L. Chang, “Helicity Amplitudes For Multiple Bremsstrahlung In Massless Nonabelian Gauge theories”, Nucl.Phys.B291, 392 (1987).
- [5] See for examples, L. J. Dixon, “Calculating Scattering Amplitudes Efficiently”, TASI Lectures 1995, hep-th/9601359; Z. Bern, L. J. Dixon, and D. A. Kosower, “Progress in One-Loop QCD Computations”, Ann. Rev. Nucl. Part. Sci. 46, 109 (1996), hep-th/9602280.
- [6] E. Witten, “Perturbative Gauge Theory as a String Theory in Twistor Space”, hep-th/0312171.
- [7] R. Penrose, “Twistor Algebra”, J. Math. Phys. 8, 345 (1967).
- [8] F. Cachazo, P. Svrcek and E. Witten, “MHV Vertices and Tree Amplitudes in Gauge Theory”, hep-th/0403047.
- [9] J. B. Wu and C. J. Zhu, MHV Vertices and Scattering Amplitudes in Gauge Theory, hep-th/0406085.
- [10] V. Khoze, “Gauge Theory Amplitudes, Scalar Graphs And Twistor Space”, hep-th/0408233.
- [11] G. Georgiou, E. W. N. Glover and V. V. Khoze, “Non-MHV Tree Amplitudes In Gauge Theory, JHEP 0407:048 (2004)”, hep-th/0407027.
- [12] D. A. Kosower, “Next-To-Maximal Helicity Violating Amplitudes In Gauge Theory”, hep-th/0406175.
- [13] J. B. Wu, C. J. Zhu, “MHV Vertices And Fermionic Scattering Amplitudes In Gauge Theory With Quarks And Gluinos”, hep-th/0406146.
- [14] I. Bena, Z. Bern and D. A. Kosower, “Twistor-Space Recursive Formulation Of Gauge-Theory Amplitudes”, hep-th/0406133.
- [15] J. B. Wu, C. J. Zhu, “MHV Vertices And Scattering Amplitudes In Gauge Theory”, JHEP 0407:032 (2004), hep-th/0406085.
- [16] G. Georgiou, V. V. Khoze, “Tree Amplitudes In Gauge Theory As Scalar MHV Diagrams”, JHEP 0405:070 (2004), hep-th/0404072.

- [17] C. J. Zhu, “The Googly Amplitudes In Gauge Theory”, JHEP 0404:032 (2004), hep-th/0403115.
- [18] A. Brandhuber, B. Spence and G. Travaglini, “One-Loop Gauge Theory Amplitudes In  $N=4$  Super Yang-Mills From MHV Vertices”, hep-th/0407214.
- [19] M. Luo and C. Wen, “One-Loop Maximal Helicity Violating Amplitudes in  $N = 4$  Super Yang-Mills Theories”, hep-th/0410045.
- [20] F. Cachazo, P. Svrcek and E. Witten, “Gauge Theory Amplitudes In Twistor Space And Holomorphic Anomaly”, hep-th/0409245.
- [21] F. Cachazo, P. Svrcek, E. Witten, “Twistor Space Structure Of One-Loop Amplitudes In Gauge Theory”, hep-th/0406177.
- [22] I. Bena, Z. Bern, D. A. Kosower, R. Roiban. “Loops in Twistor Space”, hep-th/0410054.
- [23] F. Cachazo, “Holomorphic Anomaly of Unitary Cuts and One-Loop Gauge Theory Amplitudes”, hep-th/0410077.
- [24] R. Roiban, M. Spradlin, and A. Volovich, “A Googly Amplitude from the B-model in Twistor Space”, JHEP 0404:012 (2004), hep-th/0402016.
- [25] R. Roiban, A. Volovich, “All Googly Amplitudes from the B-model in Twistor Space”, hep-th/0402121.
- [26] R. Roiban, M. Spradlin, and A. Volovich, “On the Tree-Level S-Matrix of Yang-Mills Theory”, Phys.Rev. D70, 026009 (2004), hep-th/0403190.
- [27] V. Nair, “A Current Algebra For Some Gauge Theory Amplitudes”, Phys. Lett. B214, 215 (1988).
- [28] F. Chen, M. Luo and C. Wen, in preparation.